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Realism in Mathematics: The Case of the Hyperreals

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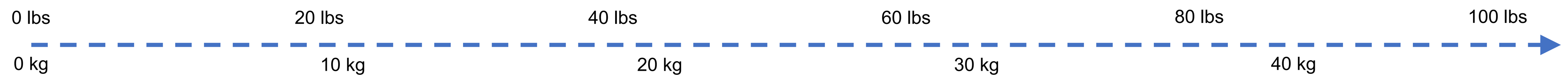
Types of realism:

Mathematical Platonism – mathematical objects exist as abstract entities independent of the physical world

Scientific realism – the terms in scientific theories refer to real physical entities and are not just useful for empirical prediction

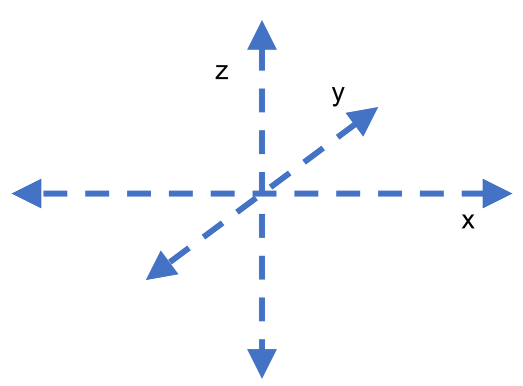
Our argument does not raise worries about Platonism or related philosophical views of mathematics – we just argue that some mathematical structures (the "hyperrreals") can't be used in a scientific realist theory. They are still good mathematics, and the standard real consequences of them are empirically meaningful.

Use of real numbers to measure mass

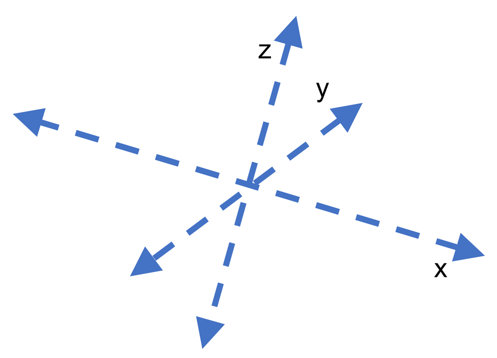


There is an arbitrary convention that combination of two objects is numerically represented by addition.
There is a further arbitrary selection of the unit – kg or lb?
Once these two arbitrary choices are made, every real number has a meaning.
Archimedean Principle: For any two objects, some finite number of copies of one adds up to a more massive object than the other
So there is a real number for the mass of every object.

Use of R^3 to represent physical space



There is an arbitrary convention that concatenation of collinear distances is represented by addition.
There is a further arbitrary selection of the origin.
Three more arbitrary selections of directions of the axes.
One last arbitrary selection of unit – mile or kilometer or parsec?
Once these **finitely many** arbitrary choices are made, every triple of real numbers has a meaning.
Archimedean Principle: "The journey of any arbitrary distance can begin with a single step."
So every point in space is represented with a triple of real numbers.



Use of R(a,b,c,d…) to represent a non-Archimedean value system

Classical economics assume moral value satisfies the Archimedean Principle – there is some number of dollars that is the value of a human life.
($7.9 million according to FDA; $9.1 million according to EPA)
But even if we reject this principle, we can do a version of the same.

5,000,000 deaths + $10 billion. = 5 megadeaths + €8.44 billion.
a = death; b = USD a = megadeath; b = EUR

Say two values are "commensurable" if a finite number of copies of either adds up to more than the other.
Assume there are finitely many commensurability classes, make one arbitrary choice of unit (a,b,c,d, etc.) for each.
Then every value has a unique representation in this field, and every element of this field has a specific meaning.

Construction of the hyperreals

Start with the real numbers.
Consider the set of all infinite sequences of real numbers.
Define operations on these sequences term-by-term.
Say that (x1,x2,x3,…)<(y1,y2,y3,…)
iff
"most" of the terms bear the < relation, where "most" is defined by an ultrafilter.
Łos's Theorem – this new structure satisfies all the same sentences as the standard real numbers.
The sequence (1, ½, 1/3, ¼, …) is "infinitesimally small".

Note – this construction requires the Axiom of Choice
(for the existence of an ultrafilter).

It allows the Transfer Principle – a sentence involving some parameters is true in the standard reals iff the same sentence with the same parameters is true in the hyperreals.

Hyperreal probabilities:
Bernstein and Wattenberg, 1969, "Nonstandard measure theory"
David Lewis, 1980, "A Subjectivist's Guide to Objective Chance"
Brian Skyrms, 1980, *Causal Necessity*

Many uses of the hyperreals:
Kanovei, Katz, and Mormann, 2013, "Tools, Objects, and Chimeras"
Bascelli et al, 2014, "Fermat, Leibniz, Euler, and the gang"

Use of the hyperreals

Problem: The hyperreals are not "rigid".
No finite number of fixed elements prevents further automorphisms.
For any set of conventional choices, there are still elements of the mathematical structure with no empirical meaning
Thus, they cannot be used with the same sort of meaning as these other mathematical structures

Statement of the constructive content

The Transfer Principle allows us to use a single quantifier to represent continuity:

$$\forall x(x \sim 0 \rightarrow f(x) \sim f(0))$$

rather than the classical definition with three quantifiers:

$$\forall \varepsilon \exists \delta \forall x(x < \delta \rightarrow |f(x) - f(0)| < \varepsilon)$$

Sanders, 2016, "The computational content of nonstandard analysis"
Bartha and Hitchcock, 1999, "The Shooting Room and conditionalizing on measurably challenged sets"

Sidebar about the Axiom of Choice:
The non-rigidity seems connected to the dependence on the Axiom of Choice.
Bascelli et al., 2014, claim that we shouldn't deny the applicability of theorems depending on the Axiom of Choice in physics or economics
We suspect that this dependence can always be eliminated for any real application.

Velupillai, 2014, "Constructive and computable Hahn-Banach theorems for the (second) fundamental theorem of welfare economics"